Achievable

AP CALCULUS BC DUMP SHEET

Unit 10: Taylor and power series

Taylor polynomials

- Degree n, centered at $x = a$: $P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$
- A **Maclaurin** polynomial is centered at 0. P_1 is the tangent line
- The coefficient of $(x - a)^n$ is $\frac{f^{(n)}(a)}{n!}$, so $f^{(n)}(a) = n!\times$ that coefficient
- The **Taylor series** is the whole sum, $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$; a Taylor polynomial is a partial sum of it

Error bounds

- **Lagrange:** $|f(x) - P_n(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1}$, where $|f^{(n+1)}| \le M$ everywhere between a and x
- Note the $n + 1$: a degree-3 polynomial needs a bound on $|f^{(4)}|$
- Smallest degree for an error below a target on an interval: use the largest $|x - a|$ and an M for the whole interval, then try $n = 1, 2, 3, \ldots$ until the bound is small enough
- If the series at x alternates, with term sizes decreasing to 0, the error is at most the size of the first term left out (the alternating series bound)

Building new series

- **Substitute:** $\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \cdots$; $\frac{1}{1 + x^2} = 1 - x^2 + x^4 - \cdots$
- **Multiply** by a power of x, term by term
- **Differentiate or integrate** term by term: the radius stays, the endpoints must be tested again. After integrating, find C at $x = a$
- **Recognize a sum:** $1 - \frac{1}{3!} + \frac{1}{5!} - \cdots = \sin 1$

Maclaurin series to know

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, for all x
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, for all x
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, for all x
- $\frac{1}{1 - x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$, for $-1 < x < 1$: geometric

Built from the geometric series by integrating:

- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, for $-1 < x \le 1$
- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, for $-1 \le x \le 1$

Power series

- $\sum_{n=0}^{\infty} c_n(x - a)^n$ always converges at its center $x = a$
- **Radius:** set the ratio test's limit below 1 and solve for $|x - a| < R$
- **Interval:** then test each endpoint $a - R$ and $a + R$ on its own, with a series test. Endpoints are the only place it can converge conditionally
- A ratio limit of 0 for every x: $R = \infty$. Infinite for every $x \neq a$: $R = 0$, only the center