Achievable

AP CALCULUS BC DUMP SHEET

Unit 10: Series and convergence tests

Sequences and series

- A sequence $\{a_n\}$ converges if $\lim_{n\to\infty} a_n$ is a finite number
- Growth as $n \to \infty$: $\ln n \ll n^p \ll b^n \ll n! \ll n^n$ ($p > 0$, $b > 1$)
- A series $\sum a_n$ converges if its partial sums $S_k = a_1 + \cdots + a_k$ approach a finite number

Geometric series

- $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$ if $|r| < 1$; it diverges if $|r| \geq 1$
- a is the first term, at the starting index: $\sum_{n=2}^{\infty} \dfrac{1}{2^n}$ starts at $\frac{1}{4}$
- Geometric: n in the exponent. A p-series: n in the base

Absolute and conditional convergence

- If $\sum |a_n|$ converges, so does $\sum a_n$: it **converges absolutely**
- If $\sum |a_n|$ diverges but $\sum a_n$ converges, it **converges conditionally**: $\sum \dfrac{(-1)^n}{n}$
- Test $\sum |a_n|$ first; if it diverges, try the AST on $\sum a_n$

Alternating series error bound

- For a series that passes the AST, its terms decreasing from a_{n+1} on, with sum S: $|S - S_n| \leq a_{n+1}$, the first term left out
- So S lies between $S_n - a_{n+1}$ and $S_n + a_{n+1}$
- Terms needed for an error below ε: solve $a_{n+1} < \varepsilon$

Convergence tests

- **nth term test:** if $\lim_{n\to\infty} a_n \neq 0$, or it does not exist, the series diverges. If the limit is 0, no conclusion: it never proves convergence
- **p-series** $\sum \dfrac{1}{n^p}$: converges if $p > 1$, diverges if $p \leq 1$. The harmonic series ($p = 1$) diverges
- **Integral test:** if f is positive, continuous and decreasing for $x \geq k$, with $a_n = f(n)$, then $\int_k^{\infty} f(x)\,dx$ and $\sum_{n=k}^{\infty} a_n$ both converge or both diverge. The integral is not the sum
- **Direct comparison,** $0 \leq a_n \leq b_n$: $\sum b_n$ converges, so $\sum a_n$ does; $\sum a_n$ diverges, so $\sum b_n$ does. Otherwise no conclusion
- **Limit comparison,** $a_n, b_n > 0$: if $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$ with $0 < L < \infty$, both converge or both diverge. Choose b_n from the dominant terms
- **Alternating series test (AST)** for $\sum (-1)^n a_n$, $a_n > 0$: converges if a_n is decreasing (eventually) and $\lim_{n\to\infty} a_n = 0$
- **Ratio test:** $L = \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$. $L < 1$: converges absolutely. $L > 1$: diverges. $L = 1$: no conclusion

Which test?

The terms have	Try
Any form: check first	nth term test
A constant ratio, n in the exponent	Geometric
$\dfrac{1}{n^p}$	p-series
Factorials or exponentials	Ratio test
Polynomials or roots	Limit comparison
$(-1)^n$	$\sum \lvert a_n \rvert$ first, then the AST
An $f(n)$ easy to integrate	Integral test

- The ratio test gives $L = 1$ on purely algebraic terms. $(n+1)! = (n+1) \cdot n!$
- Verify decreasing with a derivative, or by comparing consecutive terms