Achievable

AP CALCULUS BC DUMP SHEET

Unit 9: Parametric, polar, and vector-valued functions

Parametric equations

- $x = x(t)$, $y = y(t)$. The curve is traced in the direction of increasing t
- **Slope:** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, where $\dfrac{dx}{dt} \neq 0$
- **Horizontal tangent:** $\dfrac{dy}{dt} = 0$, $\dfrac{dx}{dt} \neq 0$. **Vertical:** $\dfrac{dx}{dt} = 0$, $\dfrac{dy}{dt} \neq 0$. $\frac{0}{0}$ gives no conclusion
- **Second derivative:** $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$. Divide by $\dfrac{dx}{dt}$ again
- Concave up where $\dfrac{d^2y}{dx^2} > 0$. Give the intervals in t
- **Arc length,** traced once for $a \le t \le b$:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- To eliminate t, solve one equation for it and substitute, or use $\sin^2 t + \cos^2 t = 1$

Vector-valued functions and motion

- **Position** $\mathbf{r}(t) = \langle x(t), y(t)\rangle$; **velocity** $\mathbf{v}(t) = \langle x'(t), y'(t)\rangle$; **acceleration** $\mathbf{a}(t) = \langle x''(t), y''(t)\rangle$
- **Speed** $= |\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2}$. **Distance traveled** $= \displaystyle\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$
- $x' > 0$: moving right; $y' > 0$: moving up. At rest only when $x' = 0$ and $y' = 0$ together
- Slope of the path: $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$
- **Position from velocity,** one component at a time: $x(b) = x(a) + \displaystyle\int_a^b x'(t)\, dt$, and likewise y. These integrals are the displacement

Polar coordinates

- $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$
- (r, θ), $(r, \theta + 2\pi)$ and $(-r, \theta + \pi)$ are the same point. A negative r plots through the origin, on the opposite ray
- $\tan^{-1}$ gives Quadrants I and IV only: for a point in II or III, add π
- The distance from the origin is $|r|$
- Along one ray θ, the distance between two curves is $|r_1 - r_2|$ when r_1 and r_2 have the same sign

Polar slope

- Treat $x = r\cos\theta$, $y = r\sin\theta$ as parametric in θ: $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$, with $r' = \dfrac{dr}{d\theta}$
- Horizontal tangent: $\dfrac{dy}{d\theta} = 0$, $\dfrac{dx}{d\theta} \neq 0$. Vertical: $\dfrac{dx}{d\theta} = 0$, $\dfrac{dy}{d\theta} \neq 0$. $\frac{0}{0}$ gives no conclusion, here too
- Farthest from the origin: compare $|r|$ where $\dfrac{dr}{d\theta} = 0$ and at the endpoints
- $\dfrac{dr}{dt} = \dfrac{dr}{d\theta} \cdot \dfrac{d\theta}{dt}$. Moving away from the origin when r and $\dfrac{dr}{dt}$ have the same sign; toward it when the signs differ

Polar area

- **One curve:** $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\, d\theta$, sweeping from $\theta = \alpha$ to $\theta = \beta$ in radians, tracing the region once ($r = \sin 3\theta$ traces its petals twice on $[0, 2\pi]$)
- A loop through the origin runs between consecutive solutions of $r = 0$. A limaçon's inner loop is where r has the opposite sign from its outer loop: $r < 0$ for $r = 1 - 2\cos\theta$
- **Between two curves:** $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta \left(r_{\text{outer}}^2 - r_{\text{inner}}^2\right) d\theta$
- **Inside both:** split at the angle where the boundary switches curves, and add $\dfrac{1}{2}\displaystyle\int r^2\, d\theta$ for each piece, using the curve nearer the origin
- Intersections: solve $r_1 = r_2$, then check the origin and any point one curve reaches with a negative r, at $\theta + \pi$